Derivation of the Laplace-Operator: Derivation of Coordinates by Partial Derivative

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Abstract :

The Laplace operator is a second order differential operator often used in theoretical physics applications. We wish to find a method to derive coordinates by partial derivative using the Laplace operator. I will discuss curvelinear coordination in the following chapters :

1- Cartesian Coordinates (x , y , z)
2- Polarity Coordinates (r, θ)
3- Cylindrical Coordinates (ρ , ϕ , z)
4- Spherical Coordinates (r , θ, ϕ)
5- Parabolic Coordinates (u, v , θ)
6- Parabolic Cylindrical Coordinates (u , v , z)
7- Curvilinear Coordinates, this general coordination

And we can use this coordination to derive more Laplace operators in any coordinates. The question is, why is the Laplace operator used in more application in physics electricity, and in wave functions[1] , for example,

A wave function can be written,

$$\psi = \psi(x + ct) + \psi(x - ct)$$

Now we can derive the wave function,

$$\frac{\partial \psi}{\partial x} = \psi(x + ct) + \psi(x - ct) \rightarrow \frac{\partial \psi}{\partial t} = c(\psi(x + ct) - \psi(x - ct))$$

$$\frac{\partial^2 \psi}{\partial x^2} = \psi(x + ct) + \psi(x - ct) \rightarrow \frac{\partial^2 \psi}{\partial t^2} = c^2(\psi(x + ct) + \psi(x - ct))$$

The last equation can give,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

In x,y,z coordinates,

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=0$$

where the Laplace operator gives in 3D x,y,z :

$$\nabla^2=\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial y^2}$$

In the quantum mechanical and Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t}=[-\frac{\hbar^2}{2m}\nabla^2+V(x)]\psi$$

And we can write d'Alembertian in spacetime :

$$\Box\equiv\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

some propertities of crul:

$$1-\nabla\otimes(kA)=k(\nabla\otimes A)$$

$$2-\nabla\otimes(fA)=f(\nabla\otimes A)+\nabla f\otimes A$$

$$3-\nabla\otimes(\frac{A}{f})=\frac{f(\nabla\otimes A)-\nabla f\otimes A}{f^2}$$

And the properties of the divergence:

$$1-\nabla\cdot(kA)=k(\nabla\cdot A)$$

$$2-\nabla\cdot(fA)=f(\nabla\cdot A)+\nabla f\cdot A$$

$$3-\nabla\cdot(\frac{A}{f})=\frac{f(\nabla\cdot A)-\nabla f\cdot A}{f^2}$$

$$4-\nabla\cdot(A\otimes B)=B\cdot(\nabla\otimes A)-A\cdot(\nabla\otimes B)$$

What the Laplace operator in Cartestian coordination

The ∇ is given by

$$\nabla=\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k$$

And the $\nabla^2\psi=\nabla\cdot\nabla\psi$

$$\nabla\cdot\nabla\psi=(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k)\cdot(\frac{\partial\psi}{\partial x}i+\frac{\partial\psi}{\partial y}j+\frac{\partial\psi}{\partial z}k)=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}$$

Laplace operator derivative by partial derivation:
The Laplace operator is given by $\nabla^2 V$ where V is the function in x,y
I will assume the function V in x,y, derivative of Laplace in polar coordinates (r, θ)

$$V = V(x, y) \rightarrow x = r\cos\theta \rightarrow y = r\sin\theta$$

By partial derivation,

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial V}{\partial r} = \cos\theta\frac{\partial V}{\partial x} + \sin\theta\frac{\partial V}{\partial y}$$

And the same in θ,

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial V}{\partial \theta} = -r\sin\theta\frac{\partial V}{\partial x} + r\cos\theta\frac{\partial V}{\partial y}\frac{\partial y}{\partial r}$$

We can solve the 2 equations by matrix operator or any methods given that,

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta} \Rightarrow \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

and in they dimension

$$\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta} \Rightarrow \frac{\partial}{\partial x} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2 V}{\partial x^2} = (\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta})\square(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta})$$

$$\frac{\partial^2 V}{\partial x^2} = (\cos\theta\frac{\partial}{\partial r}(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}))$$

$$\frac{\partial^2 V}{\partial x^2} = (\cos\theta\frac{\partial}{\partial r}\cos\theta\frac{\partial V}{\partial r} - \cos\theta\frac{\partial}{\partial r}\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\cos\theta\frac{\partial V}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}$$

$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta \frac{\partial^2 V}{\partial r^2} + 2\frac{\cos\theta \sin\theta}{r^2}\frac{\partial V}{\partial \theta} - 2\frac{\sin\theta \cos\theta}{r}\frac{\partial V}{\partial\theta\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r}$$

And in the y dimension

$$\frac{\partial^2 V}{\partial y^2} = (\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta})\square(\sin\theta \frac{\partial V}{\partial r} - \frac{\cos\theta}{r}\frac{\partial V}{\partial\theta})$$

The same algbera derivative gives,

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta \frac{\partial^2 V}{\partial r^2} - 2\frac{\cos\theta \sin\theta}{r^2}\frac{\partial V}{\partial \theta} + 2\frac{\sin\theta \cos\theta}{r}\frac{\partial V}{\partial\theta\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r}$$

By addition, gives Laplace in x,y

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}$$

Laplace in r,θ

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r}\frac{\partial^2 V}{\partial\theta^2}$$

Cylindrical Coordinates (ρ, ϕ, z)

The same method can we derive the Laplace operator in cylindrical coordinates

$$V = V(x, y, z) \to x = \rho\cos\phi \to y = \rho\sin\phi \to z = z$$

By partial derivative

$$\frac{\partial V}{\partial\rho} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial\rho} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial\rho}$$

$$\frac{\partial V}{\partial r} = \cos\phi\frac{\partial V}{\partial\rho} + \sin\theta\frac{\partial V}{\partial y}\frac{\partial y}{\partial\rho}$$

And the same in ϕ :

$$\frac{\partial V}{\partial\phi} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial\phi} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial\phi}$$

$$\frac{\partial V}{\partial\phi} = -\rho\sin\phi\frac{\partial V}{\partial x} + \rho\cos\phi\frac{\partial V}{\partial y}$$

And the same in z

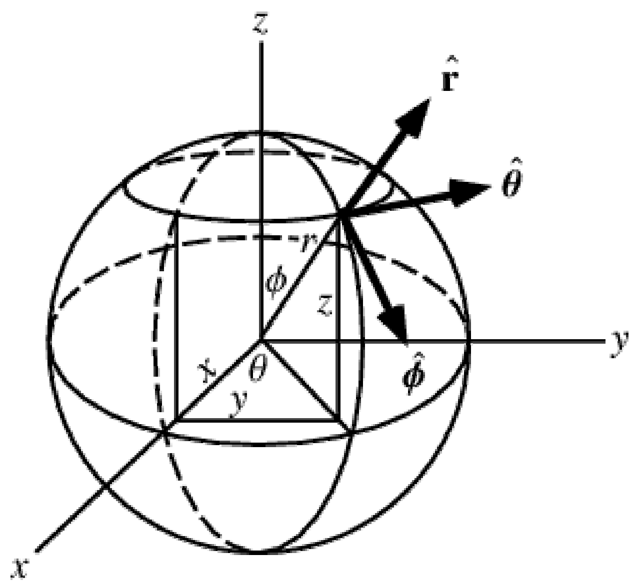
$$\frac{\partial V}{\partial z}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial V}{\partial y}\frac{\partial y}{\partial z}+\frac{\partial V}{\partial z}\frac{\partial z}{\partial z}$$

$$\frac{\partial V}{\partial z}=\frac{\partial V}{\partial z}$$

By the same method in r, θ can derivative in ρ,ϕ,z in the last give :

$$\nabla^2 V=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho})+\frac{1}{\rho}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}$$

Spherical Coordinates



Derivative by Walton 1967, Arfken 1985, it is called the Spherical Coordinates

$$\psi=\psi(x,y,z)\rightarrow x=r\cos\theta\sin\phi\rightarrow y=r\sin\theta\sin\phi\rightarrow z=r\cos\phi$$

By partial derivative

$$\frac{\partial\psi}{\partial r}=\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial\psi}{\partial r}=\cos\theta\sin\phi\frac{\partial\psi}{\partial x}+\sin\theta\sin\phi\frac{\partial\psi}{\partial y}+\cos\phi\frac{\partial\psi}{\partial y}$$

$$\frac{\partial\psi}{\partial\theta}=\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\theta}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\theta}$$

$$\frac{\partial\psi}{\partial\theta}=-r\sin\theta\sin\phi\frac{\partial\psi}{\partial x}+r\sin\theta\sin\phi\frac{\partial\psi}{\partial y}$$

$$\frac{\partial\psi}{\partial\phi}=\frac{\partial\psi}{\partial x}\frac{\partial x}{\partial\phi}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\phi}+\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial\phi}$$

$$\frac{\partial\psi}{\partial r}=r\cos\theta\cos\phi\frac{\partial\psi}{\partial x}+r\sin\theta\cos\phi\frac{\partial\psi}{\partial y}-r\sin\phi\frac{\partial\psi}{\partial y}$$

We can solve the system liner equation 1,2,3 by matrix inverse methods which give

$$\begin{pmatrix}\frac{\partial\psi}{\partial r}\\ \frac{1}{r}\frac{\partial\psi}{\partial\theta}\\ \frac{1}{r}\frac{\partial\psi}{\partial\phi}\end{pmatrix}=\begin{pmatrix}\cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi\\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0\\ \cos\theta\cos\phi & \sin\theta\sin\phi & -\sin\phi\end{pmatrix}\begin{pmatrix}\frac{\partial\psi}{\partial x}\\ \frac{\partial\psi}{\partial y}\\ \frac{\partial\psi}{\partial z}\end{pmatrix}$$

By matrix properties, gives the inverse,

$$A^{-1}=\frac{1}{DetA}\tilde{A}^{t}$$

$$A=\begin{bmatrix}\cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi\\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0\\ \cos\theta\cos\phi & \sin\theta\sin\phi & -\sin\phi\end{bmatrix}$$

By algebra solving the detriment A $DetA=-\sin\phi$ give

$$\tilde{A}^{t}=\begin{pmatrix}-\cos\theta\sin^{2}\phi & \sin\theta & -\cos\theta\sin\phi\cos\phi\\ -\sin\theta\sin^{2}\phi & -\cos\theta & -\sin\theta\sin\phi\cos\phi\\ -\sin\phi\cos\phi & 0 & \sin^{2}\phi\end{pmatrix}$$

The inverse matrix gives,

$$A^{-1} = \begin{pmatrix} \cos\theta\sin\phi & -\dfrac{\sin\theta}{\sin\phi} & \cos\theta\cos\phi \\ \sin\theta\sin\phi & \dfrac{\cos\theta}{\sin\phi} & \sin\theta\cos\phi \\ \cos\phi & 0 & \sin\phi \\ \end{pmatrix}$$

Now we can find the equation solution by matrix

$$\begin{pmatrix} \dfrac{\partial\psi}{\partial x} \\ \dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial\psi}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta\sin\phi & -\dfrac{\sin\theta}{\sin\phi} & \cos\theta\cos\phi \\ \sin\theta\sin\phi & \dfrac{\cos\theta}{\sin\phi} & \sin\theta\cos\phi \\ \cos\phi & 0 & \sin\phi \\ \end{pmatrix} \begin{pmatrix} \dfrac{\partial\psi}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ \dfrac{1}{r}\dfrac{\partial\psi}{\partial\phi} \end{pmatrix}$$

$$\frac{\partial\psi}{\partial x} = \cos\theta\sin\phi\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r\sin\phi}\frac{\partial\psi}{\partial\theta} + \frac{\cos\theta\cos\phi}{r}\frac{\partial\psi}{\partial\phi}$$

$$\frac{\partial\psi}{\partial y} = \sin\theta\sin\phi\frac{\partial\psi}{\partial r} + \frac{\cos\theta}{r\sin\phi}\frac{\partial\psi}{\partial\theta} + \frac{\sin\theta\cos\phi}{r}\frac{\partial\psi}{\partial\phi}$$

$$\frac{\partial\psi}{\partial x} = \cos\phi\frac{\partial\psi}{\partial r} - \frac{\sin\phi}{r}\frac{\partial\psi}{\partial\phi}$$

Gasiorowicz 1974, pp. 167-168; Arfken 1985, p. 108